October 5, 2022

Week 6 Notes

Logical Expressivism and the Expressively Complete Logic NM-MS

Plan:

I) **Recap**: **Three Senses of Explicit/Implicit**
II) **What is logic? Logicism and Expressivism**
III) **Rational Logical Expressivism: LX-ness**
IV) **Elaboration by Multisuccedent Sequent Calculi**
[Truncated in the middle of (IV) at end of Week 6 session.
Rest of (IV) and versions of (V) and (VI) will be for Week 7.]
V) **The Logical System NM-MS: Nonmonotonic Multisuccedent Logic**
VI) **Combining Supraclassicality and Nonmonotonicity**

A note on posting my weekly notes:
I do this because my hope that they might be useful to you in thinking through the ideas I am wrestling with outweighs my embarrassment at their sometimes chaotic character.
I appreciate that this is challenging material, and what is perhaps most valuable about it—the connections between large, potentially important philosophical ideas and detailed proposals for implementation of them—are not only intricate, but fluid and evolving.
I want to offer all the help I can.

If you do find yourself looking at them, you might think of them—as I do—as reflecting the wisdom of the advice that (to paraphrase Leonard Bernstein):
To accomplish anything of importance requires three things:
an idea, a plan for working it out, and not quite enough time.

This week we are making the transition between the two halves of the book ms. that is the source of the material I am presenting in this seminar: from *Reasons for Logic* to *Logic for Reasons*, foreshadowed in the title of my earlier article, "From Logical Expressivism to Expressivist Logics."
(The semantic bit of the project, soon to come, is not mentioned in these titles.)

This means we are crossing the boundary from a philosophical account of what we should ideally want from a logic to an account of a logic that satisfies those criteria of adequacy.

This is moving from the part of the project where I feel I am on reasonably solid ground to the part that results from Ulf and Dan's work, where they moved beyond what I was capable of and into territory where I am mostly an admiring spectator.
I will hope to bring you to be able to appreciate what they have done as I do.

But because my grip is shakier here, Dan will visit the seminar next week to present crucial features of his logic, and to answer questions.

Then the following week, Ulf will visit to introduce his semantic discoveries and constructions.

And then after that, Dan will come back to introduce his implication-space semantics—the answer to the fevered dreams of inferentialist like me for many decades now.

I. Recap. Three Senses of Explicit/Implicit: Commitment, Content, and Reason Relations

Last time we looked at **the two kinds of global structural princi**ples logicians have thought of as governing reason relations, paradigmatically consequence relations or implications.
These are **monotonicity** principles and **transitivity** principles, which together codify the relation of rational (Tarski officially is addressing specifically *logical*) consequence or implication.
They correspond to Tarski's *closure* conditions, requiring consequence to be a topological closure operator:
Monotonicity (Tarski): $X \subseteq Y \Rightarrow Con(X) \subseteq Con(Y)$.
Transitivity (Tarski): $Con(Con(X)) = Con(X)$.
We saw there are **multiple grades** of monotonicity and transitivity, and that they interact.

- In particular, **stronger forms of transitivity can promote weaker forms of monotonicity** into having the effect of stronger forms of monotonicity.

Also, though I did not talk about this last week, the effects run in the other direction as well.

- In the context of the highest grade of monotonicity, MO, the difference between Mixed Context Cut and Shared Context Cut (CT) vanishes, and even Weak Cumulative Transitivity (WCT), which requires shared context and allows cutting only on *persistent* consequences has the same effect as Mixed Context Cut.

This is all a matter of the fine structure of closure structure, when we distinguish monotonicity from transitivity, and look at different grades of each.

I suggested that a particularly illuminating lens through which to view closure structures is the operation I called "**rational explicitation**."
It is defined on sequents—so, in terms of reason relations.

We are interested in the relation between the reason relations involving two premise-sets: the premises of the base sequent and the premises that result by adding to that original base premise-set some of the consequences implied by the original.

The operative metaphor driving this idiom (vocabulary, way of talking and thinking) is that the elements of a premise-set are *explicitly* contained in it (as members of a set), and so are its *explicit* content. Its consequences, what is *implied* by it, then count as its *implicit* content, as what is contained *implicitly* in it, just as the etymology suggests.
Moving a sentence from the right hand side (the conclusion side) to the left hand side (the premise side) of a sequent then counts as *explicitating* it: making *explicit* what was *implicit* in that premise-set.

I suggested we think about structural principles in terms of the effects of explicitation in this sense.

Thought of that way, expressed in that metavocabulary for reason relations, the two most important versions of monotonicity and transitivity principles are:

Cautious Monotonicity (CM): $\dfrac{X|\sim A \quad X|\sim B}{X,A|\sim B.}$

Cumulative Transitivity (CT): $\dfrac{X|\sim A \quad X,A|\sim B}{X|\sim B.}$ (also called "Shared Context Cut")

In these forms,

the (cautious) **monotonicity** principle says that **explicitation never *loses*** or *subtracts* implicit content and

the (shared content) **transitivity** principle says that **explicitation never *gains*** or *adds* implicit content.

Together, they say that rational ***explicitation is inconsequential***, in the sense that adding implicit consequences as explicit premises never affects in any way the implicit consequences of a premise set.

And I argued that rational explicitation can often have important consequences.

Therefore we should aim for a metavocabulary (eventually: a *logic*) that is expressively powerful enough to let us talk about those effects.

This means not imposing the *closure* structure of monotonicity and transitivity (CM and CT). Rather, we need a metavocabulary that is sufficiently expressively flexible to enable us to investigate structurally *open* reason relations—relations that are *sub*structural relative to the traditional *closure* structure of global monotonicity and transitivity.

We saw that ***rational*** **explicitation** in this sense, explicitation as defined on reason relations as codified in sequents, corresponds to ***pragmatically*** **explicit and implicit normative deontic statuses** as they show up in our *pragmatic* definition of reason relations.

Implication: X|~A iff *commitment to accept* all of X precludes *entitlement to reject* A (in the multisuccedent case, all of a set Y).

Incompatibility: X#A iff *commitment to accept* all of X precludes *entitlement to accept* A (in the multisuccedent case, all of a set Y).

We then can say that if one is *explicitly* precluded from *entitlement to reject* A, one is *implicitly committed to accept* A, in the sense that if one takes a stand at all, that is the only one open. (Corresponding remarks apply to incompatibility.)

That is just to say that **a consequence that is *implied* by a premise-set, and so is *rationally implicit* in it, is what one is *implicitly* committed to accept by being *explicitly* committed to accept the premise-set**.

That is the correlation between the *content* that is ***rationally implicit*** in an explicit premise-set and the *commitment* (to accept) that is ***pragmatically implicit*** in explicit commitment (to accept) a premise-set.

Today we are going to look at a *third* variety of explicitness/implicitness: the sense in which specifically *logical* vocabularies let us make explicit *reason relations* of implication and incompatibility.
Here the key notion of explicitness is formulability as the *content* expressed by a *sentence*.

A *sentence* is what can stand to other sentences in reason relations of implication and incompatibility, both as premise and as conclusion, according to some vocabulary.

According to our pragmatic account of discursive practices, such sentences express *claimables*, what can be accepted or rejected (practical doxastic attitudes), asserted or denied (speech acts), what one can be committed or entitled to (normative deontic statuses).

Sentence express claimables that can accordingly be offered as reasons for or against other claimables in defenses of and challenges to those other claimables, and are what can be defended or challenged by reasons for or against them.

Our *semantic* claim (on down the line, in a couple of weeks) will be that it is playing the role they do in reason relations that makes sentences express the contents they do—and what identifies those contents as specifically *propositional* contents.

The claim to be introduced today is that it is the particular and characteristic task of *logical* vocabularies to *make explicit* in claimable sentential form the reason relations in virtue of which anything is a claimable at all.
That is looking *forward*.

Looking *back* briefly once more,
I ended last time pointing out that one metaconceptual payoff from not imposing closure structure on reason relations—opening up structurally so as to be able appropriately to distinguish and keep separate sets of books on *implicit* and *explicit* content—is that we can bring into view the possibility of sets of premises (sentences of a vocabulary) that are *explicitly* coherent, but *implicitly in*coherent, in the sense that they can be *made explicitly* incoherent by explicitation of some of their consequences. Dually, we can consider the possibility of sets of premises that are *explicitly in*coherent, but *implicitly* coherent, in that they can be made explicitly coherent by explicitation of some of their consequences.

This is a more rarified effect than the mere nonmonotonicity of incoherence-incompatibility, which underwrites the possibility of explicitly incoherent premise-sets having coherent supersets.

(All explicitly coherent premise-sets have supersets that are *in*coherent, because we require that the whole lexicon, comprising *all* the sentences of a vocabulary, be incoherent—to ensure that there are incompatibilities in every vocabulary.)

We will see in our logic important examples of another rarified phenomenon that is only visible in open-structured reason relations. This point is downstream of my suggestion that the classical principle of *explosion* of *inconsistent* premise-sets (whether explicitly or only implicitly inconsistent)—the idea that they imply everything—might be restricted to *persistently* incoherent premise-sets, which would include explicitly *inconsistent* sets, once we have negation in the vocabulary.

This is *explicitly*, but not *persistently* incoherent sets being made explicitly *and* persistently incoherent by explicitation, when we look at the effects of making *implied* negations explicit as premises.

II. What is logic? Logicism and Expressivism

Nearly every philosopher of logic will have their own answer to this question.
I'll mention a few popular ones, contrasting it with our idiosyncratic one.
I think the two most important subquestions are:

a) Demarcation Question:
What is the distinctive role characteristic of specifically *logical* vocabulary and the concepts such vocbulary expresses?
The answer to this question will guide us in deciding whether various candidate bits of vocabulary deserve to count as genuinely *logical*: the set-theoretic epsilon, modal operators, higher order quantifiers…).

b) Reasons Question:
What is the relation between logic and reasons?
(Logical consequence/incompatibility and consequence/incompatibility generally.)
Getting the Reasons Question right is the key to answering the Demarcation Question.
Two views:
Logicism about Reasons
Rational Expressivism about Logic.

<u>Logicism:</u>
LR says good reasons are always logically good reasons.
Logic determines and explains what good reasons are.

Suppose I claim "The streets will be wet at the end of the seminar."
You challenge that claim, and I defend it by offering as a reason:
"It is raining."
The logicist thinks that there is a suppressed premise in my argument for the claim that the streets will be wet.
(It is what old-timey logicians called "an *enthymeme*": an argument with a missing or suppressed premise.)
The logicist thinks the *real* argument goes like this:

> *If* it is raining, *then* the streets will be wet.
> It is raining.
> So:
> The streets will be wet.

In this case, the missing premise is a conditional.

The conditional is a substantive, nonlogical premise, expressed using logical vocabulary.
What makes the reasoning good is that it is an instance of the logically valid form of argument *modus ponens*, detachment from a conditional.
The ability to reason is always an *implicitly logical* capacity (there is that word again).
Even if I can't recognize logical tautologies, I must implicitly grasp what is expressed by conditionals, and be able to distinguish at least *some* basic logically good patterns in order to reason at all.
(Here I am following Sellars's diagnosis in his early—1953—essay "Inference and Meaning.")

Logic explains the distinction between good reasons and bad ones, what follows and what does not.
The restriction might be to theoretical-doxastic reasons, as opposed to *practical* reasons. Or to *deductive* reasons. If this is meant in a non-question-begging way, then I think it must be read as *commissive* or *commitment-preserving* implications, as opposed to *permissive* or entitlement-preserving implications.
The view is that talk about *rational* consequence (of restricted sort) is talk about *logical* consequence. Here is it important to many that under suitable restrictions, rational choice theoretic notion of "good reason" yields classical logic.
Here the key idea is that logic is to give us a theory of *good reasons*.
Graham Priest is a good contemporary example.

This logicist view of reasoning is not a creature of recent, mathematically sophisticated logic.
It is quite traditional.
We find it in the Early Modern idea, forwarded already by **Montaigne**, that a dog chasing a rabbit faced with a fork in the road, who carefully sniffs one path and, finding no scent, then sets off without further ado or investigation down the other path, is showing *implicit* practical grasp of the **disjunctive syllogism**:
A or B. Not A. So B.
For it is only in virtue of the *logical* goodness of that form of reasoning that the reasoning counts as *good*, and the dog who can act on such reasons *intelligent*.

The expressivist idea is that this is exactly wrong.
The conditional "If it is raining, then the streets will be wet," is not a missing, suppressed, or *implicit* premise in a logically valid deductive inference by *modus ponens*, and that logically valid form does not *make* the reasoning good or substantively *explain* its goodness.
The function of the conditional is rather an expressive one:
It lets us *say explicitly that* the original implication is a good one.
By asserting it, the asserter explicitly endorses the reason given *as* a good one.
But it is not *logically* good but, as Sellars puts it, *materially* good.

It is not good because of what the bit of *logical* vocabulary that is the conditional "if…then__" means.
What makes it good is the contents of the *non*logical concepts "raining" and "wet."

REL says logic expresses reason relations, it is a tool for talking about reasons.
Logic lets us say explicitly what follows from what and what is incompatible with what: what is a reason for and what is a reason against what.
Conditionals and negation are the big ones (for propositional logic, which is all I'll discuss in the course).
(Boolean helper-monkeys of conjunction and disjunction.)
Some other popular questions I will not, largely address:
Correctness question, paired with Demarcation question by Quine and Putnam in the '60s.
Pluralism question.
Analyticity of logic question: are logical truths true in solely in virtue of the meanings of the logical vocabulary? (Boghossian)
Exceptionalism? (Contemporary, Williamson is anti-exceptionalist.)
Adoption question.
REL makes the Correctness question vanish, by being pluralist.

So, for instance, we don't think there is a single conditional that is "correct"—not even our NM-MS conditional.
Rather, there are as many ^{s}correct^{s} conditionals as there are senses of "good implication."

- The much maligned two-valued conditional (horseshoe) codifies the sense of "good implication" in which it is a good-making feature of an implication that it does *not* have a true premise and a false conclusion. At least we can admit that that would be bad. This might be minimal, but it is marking a genuine evaluation.
- The intuitionist conditional marks the sense in which an implication is good if there is a recipe for turning a reason for (originally: proof of) the antecedent into a reason for the conclusion.
- C.I. Lewis's "hook" marking what he called "strict" implication codifies the sense in which it is a good thing if it is *impossible* that the premises be true and the conclusion false.
- And so on. The important question to ask about a conditional is not whether or not it is "correct," but what dimension of correctness or goodness of implication does it express or codify?

(Reading of various conditionals: two-valued, intuitionist, strict implication…).
But NM-MS is strictly better than all the others, because it can express open-structured reason relations as well as closed ones.
REL is exceptionalist.

REL dissolves the adoption problem.

Logicism and expressivism as answers to the Reasons Question have in common a satisfying answer to the issue of whether and **why philosophers should care about logic**.
For logicists, it is because philosophers as such care about reasons, and logic is the science of good reasons.
For expressivists, it is because philosophers as such care about reasons, and logic provides the expressive resources to make reason relations *explicit*, to bring them into the discursive practices they *implicitly* normatively govern in sentential-claimable form, so that they can be explicitly challenged and defended, offered as and stand in need of reasons themselves.

Logicism says that the *goal* of philosophy is the logical substructure from which are derived all reason relations and so norms for reasoning, and on which they are based. Expressivism says that **logic is a *tool* for philosophy**. It lets us make things explicit, say things about reason relations, and so norms for reasoning.

For expressivists, logic is the organ of rational self-consciousness—and so, we semantic inferentialists will claim, the organ of *semantic* self-consciousness. For meaning or conceptual content is, for such inferentialists, a matter of role in reason relations.

In this way, both logicist and expressivist answers to this crucial subquestion of the overarching "What is logic?" question contrast with other conceptions of logic that don't see logic as of central importance for philosophers.

In explicating this opposition of explanatory strategies, one logic-first then reasons, and the other reasons-first then logic, I should emphasize that it is meant to be *exclusive*, but it is not *exhaustive*.
There are other views about what demarcates logic, what the distinctive task of logic is, which don't fall neatly under either rubric, logicism or expressivism.

For instance, **Neil Tennant** speaks for many when he sees the defining job of logic as systematizing the implications appealed to in specifically mathematical argumentation.
This is a more modest conception, and it is true to a central strand in the logical tradition, particularly since Frege. Neil Tenant's view that the job of logic is to codify specifically the reasoning of *mathematics* can take either logicist or expressivist forms.
And Tennant's view makes clear why philosophers *of mathematics* should care about logic.
But what about everyone else?
If one does not make the logicist leap to thinking that mathematical reasoning shows what is right about reasoning in general (the leap, to be fair, that his admiration for Euclid inspired in Plato), then on this view logic is at most of specialized, sub-disciplinary significance in philosophy.

At the other end of this spectrum of domain restriction is:

The Erlangen-Tarski-Sher approach (which they associate with Frege, and which is one strand of his view) championed now by Williamson.
Beginning with Felix Klein's Erlangen Programme classifying geometries as more or less general depending on what class of symmetries they impose, this line of thought understands logic as distinguished by its topic-neutrality, or its reason relations applying to reasoning about any subject-matter area whatsoever.
Where mathematicians reason about structures and physicists' reason within *their* restricted vocabulary, *logical* reason relations apply *no matter what* the topic of discussion is, no matter what *vocabulary* one is employing.
This leads to a *substitutional* or *permutational* approach that is antithetical to what we are doing.
MacFarlane has good criticisms of this view in his dissertation on notions of the "formality" of logic.
(This is "formality as topic-neutrality" in his botanization.)
I will not discuss this line of thought, this demarcational strategy for logic, here.
It will come up again when we look at Fine's truthmaker semantics, which shows how to operationalize it.

My concern here is not to argue against logicism about reasons—though I think it is wrong—but to develop the expressivist view.

III. Rational Logical Expressivism

We can make the project of logical expressivism more precise.
In my Locke Lectures, *Between Saying and Doing*, I begin by dividing the criteria of adequacy of expressivist understandings of logical vocabulary into two parts.
Both concern the relations between *logical* vocabulary and a non- or prelogical material *base* vocabulary.
A *base vocabulary* that already includes *material* (non- or prelogical) reason relations of consequence and incompatibility is presupposed.
The partition it offers between candidate implications and incompatibilities that *do* and those that do *not* hold is explanatorily and (so, in that sense) conceptually prior to the distinction made between *logically* good implications and incompatibilities, in the logically extended vocabulary.

i) First, logical vocabulary should be ***elaborated from*** the *base* vocabulary.
ii) Second, logical vocabulary should be ***explicative of* the** base vocabulary, in the sense of making it possible to *make explicit* the reason relations of the base vocabulary by formulating sentences in the logically extended vocabulary that *say that* implications and incompatibilities hold in the base vocabulary.

As shorthand, I for this dual criterion of being *el*aborated from and *ex*plicative of, I will say that logic should be "LX" for the base vocabulary.

That is where the "rational" in "rational logical expressivism" comes in: what is expressed by logical vocabulary is *reason* relations, which govern the giving of reasons for and against, the defending and challenging of doxastic commitments (acceptances/rejections).

Both of these conditions require further explication.

The first concept we need is that of a <u>vocabulary</u> (and then, <u>metavocabulary</u>).
I suggested using this term, following Rorty, as a successor to talk of "languages" and "theories," in the light of Quine's criticisms of that distinction in "Two Dogmas of Empiricism."
I have said I would use the term elastically and capaciously, but I have not so far tried to say more precisely what I mean by it.
Now we are in a position to do that.
By "vocabulary" I shall henceforth mean a *lexicon* or set of sentences, together with a set of *reason relations* on that lexicon.
That is a set of implications and incompatibilities relating the sentences in the lexicon.
It is what last time I called a "material reason relation frame," or sometimes (bowing to earlier use by expressivists) "MSF" (which stood for "material semantic frame").
We'll see that there are many different ways to represent vocabularies in this technical sense.

Recall that when I introduced the notion of a vocabulary in my first lecture, I emphasized that we should think of vocabularies as lexical items *in use*, or *as* they are *used* in some discursive practice. How is that thought related to the formal definition I have just offered? In Week 3 I discussed how discursive practices as specified in a two-sorted deontic normative pragmatic metavocabulary first permit the *definition* of reason relations of implication and incompatibility and then can be understood as *normatively governing* the use of claimables that stand in those reason relations as assertible and deniable, rationally challengeable and defensible. Against that background, specifying vocabularies in terms of lexicons and reason relations gives them pragmatic significance for the use of the sentences in the lexicon.

Three subsidiary *desiderata* or criteria of adequacy will also be motivated:

iii) **Universality** of LX-ness:
The logical vocabulary should be LX for *every* vocabulary.
In particular, it should not be restricted to being LX only for vocabularies that satisfy structural closure principles of monotonicity and transitivity of any grade.

iv) **Conservativeness** of *elaboration*:
The lexicon and reason relations of the base vocabulary (from which the logical vocabulary is elaborated and of which it is explicative) should be contained as subsets in the lexicon and reason relations of the logically extended vocabulary.

v) **Comprehensiveness** of explication:
The logical vocabulary should be capable of explicating the reason relations not only of the original base vocabulary, but also of the logically extended vocabulary.

Universality of LX-ness is an aspiration.
But if we are going to build an expressive tool, we should *want* to build one that is as expressively powerful as possible. Maybe we can't do that. I couldn't. But it *is* possible, Ulf and Dan did it, and the story I'm embarked on here is how it works.

Conservativeness:
Here the short story is that this condition on elaboration, that it extend and so include the base lexicon and preserve the reason relations of the base vocabulary on that base lexicon, is a consequence of the goal of explication.

In *this* sense of "express explicitly," **if the point of introducing logical vocabulary is to *express* the reason relations of the base vocabulary, then introducing it should not *change* those reason relations.** It should add to them while preserving them.
Note that in this regard, logical explicitation of reason relations contrasts with rational explicitation of sentential content, which I argued should *not* be "inconsequential."

A slightly longer story goes through Prior's definition of what he called a "runabout inference ticket," namely a logical connective he called **'tonk'** that licenses arbitrary new implications involving only prelogical vocabulary. It has the usual introduction rule for

*dis*junction, permitting the transition from A to A∨B (in this case A tonk B), and the usual elimination rule for *con*junction, permitting the transition for A&B (in this case A tonk B) to B. Applying both rules permits the implication by A of arbitrary B. The Pitt logician Nuel Belnap showed that it is necessary and sufficient to avoid logical connectives that will "tonk up" your reason relations to require that the rules elaborating the reason relations governing logically complex sentences conservatively extend the reason relations on the base, in the sense of permitting no new implications involving only old, prelogical, vocabulary.
(I tell this story in Chapter 1 of *Articulating Reasons*.)

Finally,
Comprehensiveness:
For reasons that I explain in Chapter 2 of *Making It Explicit* (and Ch. 1 of *AR*), I take Frege's 1874 *Begriffsschrift*, the founding document of modern logic, to articulate an avowedly *expressivist* logical project. The task of logic is to render explicitly the reason relations governing *non*logical concepts, to begin with of arithmetic, but then also of geometry and even physics, and chemistry.
Remarkably, though it is not something Frege himself ever remarks on, Frege's logic *also* codifies its *own* reason relations. He marks this fact by appending to the original edition a table showing what later theorems each earlier theorem is appealed to as a premise in the proving.

Because the reason relations of the logical vocabulary are dependent on the base vocabulary along the two dimensions of elaboration from and explication of, logical vocabulary counts as a *meta*vocabulary. Because the reason relations of our two-sorted deontic normative pragmatic vocabulary depends on the discursive practices that are the *use* of some base vocabulary (for instance, ordinary empirical descriptive vocabulary), it, too, counts as a *meta*vocabulary. By contrast, there is no prior base vocabulary that stands to ordinary empirical descriptive vocabulary as it stands to pragmatic and logical metavocabularies. It is *just* a vocabulary. What makes logical vocabularies a distinctive *kind* of metavocabulary is also precisely its dual role as elaborated from and explicative of the reason relations of any vocabulary whatsoever.

More on **Explicitation**:

The basic idea is that it is the job of the conditional arrow → to codify in the logically extended object language what is expressed by the implication turnstile |~.
And it is the job of negation ¬ to codify in the logically extended language what is expressed by the incompatibility hash mark #.
On this expressivist view, conjunction and disjunction, though necessary, play only an auxiliary expressive role, helping us make explicit collections of premises.

Conjunction & explicates the comma on the left-hand, premise side of the multisuccedent sequent turnstile, and disjunction ∨ explicates the comma on the right-hand, conclusion side of the multisuccedent sequent turnstile.
I sometimes mark this second-class, merely auxiliary role by contrast to the first-class expressive role of the conditional and negation by derisively referring to conjunction and disjunction as "Boolean helper-monkeys."

To perform its defining expressive task of codifying implication relations in the object language, conditionals need to satisfy the
Deduction-Detachment (DD) Condition on Conditionals: **Γ|~A→B iff Γ,A|~B.**
That is, a premise-set implies a conditional just in case the result of adding the antecedent to that premise-set implies the consequent. A conditional that satisfies this equivalence can be called a "Ramsey-test conditional," since Frank Ramsey first proposed thinking of conditionals this way.

To perform its expressive task of codifying incompatibility relations in the object language, negation needs to satisfy the
Incoherence-Incompatibility (II) Condition on Negation: **Γ|~¬A iff Γ#A.**
That is, a premise-set implies not-A just in case A is incompatible with that premise-set.
We can also read Γ#A in this single-succedent formulation as saying that the set Γ∪{A} is incoherent. That will be important for the multisuccedent formulation we will be using going forward.
Notice that it follows that ¬A is the minimal incompatible of A, in the sense of being implied by everything that is incompatible with A.
Aristotelian contradictories are minimal Aristotelian contraries in just the same sense.
¬(x is green) is implied by all of the contraries of "x is green," such a s "x is red," "x is blue," and so on.

The expressivist idea is that logical vocabulary can codify reason relations explicitly in the form of logically complex sentences formed from items of the base lexicon by introducing conditionals and negation according to principles such as DD and II.

IV. Elaboration by Multisuccedent Sequent Calculi

Elaboration requires a *function* that takes base vocabularies as arguments, and yields logically extended vocabularies as unique values. Since vocabularies consist of lexicons plus reason relations, this means we need two kinds of component functions:

a) **Lexical-syntactic** elaboration, taking the base lexicon as argument and producing the logically extended lexicon as value.
b) **Reason-relational** elaboration, taking the reason relations of the base vocabulary as arguments, and yielding reason relations defined on the logically extended lexicon as values.

We will need to regiment things a little in order explicitly to define such functions.
The lexical-syntactic elaboration function f_{lex} is straightforward, but it is worth taking it a bit slow so as to be clear about how it works.
We start with a nonlogical lexicon L_0, which can be just a set of sentence letters.
We get the new lexicon, $f_{lex}(L_0) = L$ by adding to L_0 all the logically compound sentences we can form from the atoms in L_0 by forming conditionals, negations, conjunctions, and disjunctions of them.
That is, $f_{lex}(L_0) = L$ is the smallest set (by inclusion) such that:
$L_0 \subseteq L$, and if $\alpha,\beta \in L$, then $\alpha \rightarrow \beta$, $\alpha \& \beta$, $\alpha \vee \beta$, and $\neg\alpha$ are $\in L$.
(It can be shown that that defines a unique set—ironically, because $f_{lex}(L_0)$ is a topological *closure* operator)
L adds to L_0 a (countably) infinite set of logically complex sentences, each of which is only finitely long (only has a finite number of logical atoms + connectives).

Of course, it is the reason-relational elaboration function that is more challenging to define.

Here we will help ourselves to Gentzen's notational convenience that I warned last time we must not be misled by into thinking that incompatibility is a second-class reason relation relative to implication.

We *can* think of that as a triple: <L, |~, #>, with the turnstile being an implication relation and the hashmark being an incompatibility relation.
Or we might represent the second kind of reason relation by a set Inc of *incoherent* sets of sentences, from which we can derive the incompatibilities, since incompatibility is symmetric *de jure*.

I have been considering ***single succedent*** implication and incompatibility.
But the more general case is ***multi*succedent** reason relations.
Then implications are represented by sets of pairs of subsets of the lexicon L.

Bilateralism tells us how to read these: Γ|~Δ is a pair of sets of sentences such that accepting all (every sentence) of Γ and rejecting *all* (every sentence) of Δ is *out of bounds*.
In the version I have suggested, commitment to accept all of Γ precludes entitlement to reject all of Δ.

I have pointed out the dangers involved in using the convenient notational trick that lets us use just one kind of sequent, written with a turnstile, to encode not only implications, but also incompatibilities-incoherences.
Nonetheless, it *is* convenient.
And there are also deep structures that become more visible if we use it.
Being wary of being misled by this notational convenience, we can indulge in it.
Then including in the (extended) implications <Γ,∅>, with the second element being the empty set, or writing Γ|~ , with empty RHS, is to be read as saying that Γ is incoherent.
If Γ is incoherent, then any subsets of it S and T such that S∪T=Γ are incompatible: S#T.
That is, commitment to accept all of S rules out entitlement to accept all of T.
Then we can represent a vocabulary just as a pair of a lexicon L and a set $R^2 \subseteq \mathcal{P}(L) \times \mathcal{P}(L)$ of pairs of subsets of L: namely those pairs <Γ, Δ> such that Γ|~Δ.
So, henceforth a vocabulary is just a pair of a lexicon and a set of pairs of sets of elements of the lexicon: <L,R^2>.

It is in this form that we want to introduce a ***reason-relation elaboration function*** $\mathbf{f_{rat}(R^2_0) = R^2}$, that takes as arguments sets of pairs of sets of elements of a base lexicon L_0, and returns as values sets of pairs of elements of the lexicon of the logically extended language $f_{lex}(L_0) = L$.
The whole elaboration function will be the pair of the lexical-syntactic and the reason-relational elaboration functions: f = < f_{lex}, f_{rat}>.

Thought of in these terms, Gentzen's sequent calculi (paradigmatically, his LK multisuccedent sequent calculus version of classical logic) define reason-relation elaboration functions.

His sequent calculi can be thought of as having three parts:
Axioms, corresponding to reason relations in the base vocabulary,
Structural rules imposing closure by monotonicity and transitivity, and
Connective rules, which show how to elaborate reason relations in the base vocabulary into the reason relations of the logically extended vocabulary.

His structural principles, which we have seen before (ignoring what is needed to turn his lists into our sets), are

Weakening (MO):

$$\frac{\Gamma \mid\sim \Delta}{\Gamma,\Theta \mid\sim \Delta,\Pi}$$

Cut = Shared Context Transitivity (CT):

$$\frac{\Gamma, A \mid\!\sim \Delta \qquad \Gamma \mid\!\sim A, \Delta}{\Gamma \mid\!\sim \Delta}$$

We don't want to restrict ourselves to explicating the reason relations only of vocabularies with closed reason relations, and if the reason relations of the base vocabulary are open (nonmonotonic and nontransitive to any degree) imposing Gentzen's structural principles would not extend them conservatively.

For axioms, Gentzen takes all instances of
Reflexivity (RE): $p \mid\!\sim p$, for p in the nonlogical base lexicon.
This is what we call a "flat" reason relation—even when extended to the full Containment (CO) structure of all instances of $\Gamma, p \mid\!\sim p, \Delta$ (which he gets by imposing MO on these instances of RE).
We want to use *all* the reason relations of the base vocabulary as axioms, which will be elaborated by the application of the connective rules.
So we start with one axiom schema:

Axiom:

$$\frac{\Gamma \mid\!\sim_0 \Delta}{\Gamma \mid\!\sim \Delta}$$

($\Gamma \mid\!\sim_0 \Delta$ can only hold if $\Gamma, \Delta \subseteq L_0$.)

So this rule ensures that all the sequents of the base vocabulary will still be sequents in the logically extended vocabulary.
(This makes the elaboration *conservative*, a condition to which we will return while addressing the *explication* condition on specifically *logical* extension of reason relations.)

For us, *all* the work will be done by connective rules.
As an example of how sequent-calculus connective rules can elaborate reason relations, consider:

L&: $$\frac{\Gamma, A, B \mid\!\sim \Theta}{\Gamma, A\&B \mid\!\sim \Theta}$$

R&: $$\frac{\Gamma \mid\!\sim A, \Theta \qquad \Gamma \mid\!\sim B, \Theta}{\Gamma \mid\!\sim A\&B, \Theta}$$

Suppose that is the *only* logical connective.
First, we need to define the *lexical* extension function $f_{lex}(L_0) = L$.
We define $L_{\&}$ as the smallest set (by inclusion) such that:

Axiom_{lex}:	$A \in L_0$	$\Rightarrow \quad A \in L$.
a.	$(A \in L \text{ and } B \in L)$	$\Rightarrow \quad A\&B \in L$.

This extends the logically atomic base lexicon to include all conjunctions, including conjunctions of conjunctions with atoms, conjunctions of conjunctions….

To define $f_{rat}(R^2_0) = R^2$, the elaboration function for reason relations, we define R^2 as the smallest (by inclusion) set such that:

1) Axiom: $<\Gamma,\Delta> \in R^2_0 \Rightarrow <\Gamma,\Delta> \in R^2$
2) L&: $<\Gamma\cup\{A\}\cup\{B\}, \Theta> \in R^2 \Rightarrow <\Gamma\cup\{A\&B\}, \Theta> \in R^2$
3) R&: $(<\Gamma,\{A\}\cup\Theta> \in R^2$ and $<\Gamma,\{B\}\cup\Theta> \in R^2) \Rightarrow <\Gamma, \{A\&B\}\cup\Theta> \in R^2$

These three rules tell us:

(1) Which sequents (pairs of sets of sentences, representing premises and conclusions of reason relations) from the base vocabulary are in the &-extended vocabulary—all of them. And
(2) Which sequents containing conjunctions in their premise-sets are in the &-extended vocabulary. And
(3) Which sequents containing conjunctions in their conclusion-sets are in the &-extended vocabulary.

We are going to do the same thing with all the connective rules of NM-MS.
The only thing that changes with more connectives is that we have to make R^2 the smallest set that includes the result of applying *all* the rules to the base vocabulary $<L_0, R^2_0>$.

What we are doing in both syntactic and reason-relations case **is *closing* the base under a set of rules**. It is closed because f(f(base)) = f(base) and base1$\subseteq$base2 $\Rightarrow$ f(base1)$\subseteq$f(base2), that is transitivity and monotonicity hold of these elaboration functions.
We need those nice properties, but we need *not* to have the reason relations force to be transitive and monotonic because *these* relations are.
And that is what we get.
Still, there is a certain irony in using closure functions to elaborate non-closed bases into non-closed extensions, if we think about the reason relations.